Classification of Topological Phases of Matter

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(Dated: March 2020)

The past forty years have witnessed the discovery of a whole new class of quantum materials, with some unique properties, such as long-range entanglement and a topology-dependent ground-state degeneracy. Classifying those so-called topological phases of matter is essential to understand their properties and find potential applications. While a large range of simple topological systems have already been classified, more exotic systems continue to resist classification attempts. After having introduced the two main paradigms to classify phases of matter—Landau's symmetry-breaking and topological order—, we go on to describe the recent achievements in terms of classification and the remaining challenges. We then explore more recent discoveries that go beyond topological order—the fracton order—and the different attempts at classifying them.

I. INTRODUCTION

An important objective of condensed matter physics is to allow the discovery and efficient design of new materials, in phases where quantum effects play a critical role. Among the most sought-after quantum materials, we can find high-temperature superconductors, quantum hard-drives, noise-free quantum computers, and spintronic devices, which all have a wide range of far-reaching applications. All of those materials can be studied with statistical physics models—typically lattices of spins with some particular interactions—and the desired effects reproduced mathematically. However, two challenges remain to implement those materials in practice: one is theoretical—what is the set of lattice models that exhibit a certain quantum effect, and for what range of parameters?—and the other is experimental—how can we synthesize a material that accurately reproduces one of those models?

To solve the theoretical challenge, a first step would be to build a periodic table of all quantum materials, where one could fully characterize a model from the lattice type and dimension, the nature of the interactions and their internal symmetries. Conversely, an experimentalist seeking a particular quantum effect could find all the models and phases that exhibit this effect in the table.

A first classification of this kind was achieved by Lev Landau in the middle of the 20th century, using the notion of symmetry-breaking. While his classification was completely successful at predicting the properties of most phases of matter—such as ferromagnetism—, new discoveries in the 1980s showed the incompleteness of his framework. Phenomena such as the quantum Hall effect and materials such as quantum spin liquids could not have been predicted using this paradigm. Since those phenomena are driven by topological effects—properties that are invariant to local perturbations and depend on the lattice topology—, they were called topological phases of matter. Following their identification as a new type of matter in 1989 [1], many progress have been made to classify them exhaustively. In particular, one and two-dimensional systems are almost fully classified, as well as systems of non-interacting bosons and fermions. However, many challenges remain in higher dimensions and for more complex systems, such as the recently discovered fractons.

In this paper, we will review the main ideas behind topological phases of matter, starting from Landau's symmetry-breaking paradigm, contrasting it to the notion of topological order, and finally presenting the current classification results in terms of topological order and beyond.

II. LANDAU'S SYMMETRY-BREAKING PARADIGM

Before the discovery of the quantum Hall effect, spin liquids and other new forms of matter in the 1980s, it was widely believed that all the phases of matter could be classified through a unique framework: the symmetry-breaking paradigm. The basic idea is that any material can be described by a Hamiltonian, which characterizes the interactions between the particles and the effects of the environment. This Hamiltonian almost always have some symmetries—parity, time reversal, translation invariance, etc.—that often influence the symmetries of the ground-state. When the ground-state has the same symmetries as the Hamiltonian, we are in a symmetric phase, otherwise we are in a symmetry-breaking phase.

The prototypical example of symmetry-breaking is the transverse-field Ising model, described by the following Hamiltonian:

$$H(g) = -\sum_{i,j} Z_i Z_j - g \sum_i X_i \tag{1}$$

where Z_i and X_i are the corresponding Pauli matrices applied to the spin i. This Hamiltonian is characterized by a global $\mathbb{Z}_2$-symmetry: when all the spins are flipped at once, the Hamiltonian stays the same, or more formally

$$\left[H, X^{\otimes n}\right] = 0 \tag{2}$$

We can study the ground state of this Hamiltonian in two regimes: $g \to \infty$ and $g = 0$.

When $g \to \infty$, the Hamiltonian becomes

$$H = -g \sum_i X_i \tag{3}$$

Its ground state is given by the equal superposition of all the spin configurations:

$$|\psi_0\rangle = \frac{1}{\sqrt{d}} \sum_{\text{all configurations}} |0110...\rangle \tag{4}$$

This phase is symmetric, since flipping all the spins at once does not change the ground-state:

$$X^{\otimes n} |\psi_0\rangle = |\psi_0\rangle \tag{5}$$

On the other hand, when $g = 0$, the Hamiltonian becomes

$$H = -\sum_{i,j} Z_i Z_j \tag{6}$$

which has 2-dimensional ground-space given by

$$|\psi_0\rangle = a\,|0...0\rangle + b\,|1...1\rangle \tag{7}$$

where $|a|^2 + |b|^2 = 1$. In general,

$$X^{\otimes n} |\psi_0\rangle \neq |\psi_0\rangle \tag{8}$$

meaning that this phase is symmetry-breaking.

Those phases can be classified the following way. Each phase is labelled as (G_H, G_ψ) where G_H is the symmetry group of the Hamiltonian and $G_\psi \subseteq G_H$ the symmetry group of the ground state. In our example, the symmetry group of the Hamiltonian is given by $G_H = \{X^{\otimes n}, \mathrm{I}\}$, the symmetric phase corresponds to $G_\psi = G_H$ and the symmetry-breaking phase to $G_\psi = \{\mathrm{I}\}$.

Finally, a good classification of phases should be physical, meaning that each phase can be characterized by some distinguishable properties that can be measured in the lab. In the Landau paradigm, this distinguishable property is called the **order parameter** and corresponds to an observable m such that $\langle m \rangle = 0$ in the symmetric phases and $\langle m \rangle \neq 0$ in the symmetry-breaking phase. In the Ising model, m is the magnetization, defined as the average value of the spins:

$$m = \frac{1}{N} \sum_i Z_i \tag{9}$$

Therefore, the Landau symmetry-breaking paradigm seems to give a very general method to classify phases with group theory and characterize them with a local order parameter.

III. TOPOLOGICAL ORDER

Now we are allowed
To disavow Landau
Wow...
—John Preskill

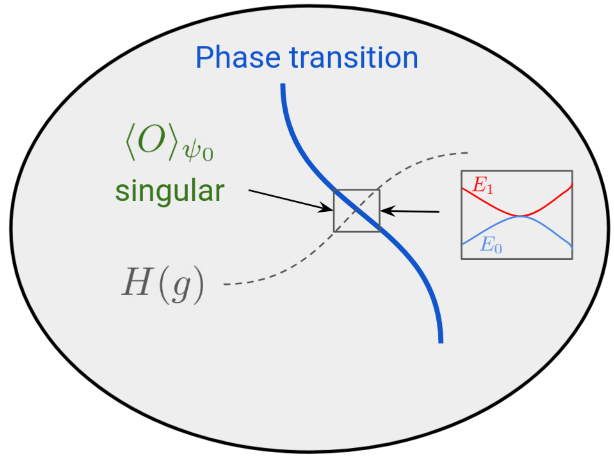


FIG. 1. Schematic illustration of a phase transition. The grey area represents a set $\mathcal{H}$ of Hamiltonians. A path of Hamiltonians $H(g) \in \mathcal{H}$ goes through phase transition if there exists an observable that become singular along the path, or equivalently that the energy gap has closed

A. Beyond Landau's paradigm

In the 1980s, many new quantum phenomena started to appear that could not be explained with traditional methods: the quantum Hall effect [2, 3], high-temperature superconductivity [4], quantum spin liquids [5], etc. It is to explain a particular instance of the latter—the chiral spin liquids—that Xiao-Gang Wen proposed the theory of **topological order** [1]. Indeed, people first identified the chiral spin liquid phase as a symmetry-breaking phase (for time-reversal and parity). However, it was then realized that different macroscopic properties (such as the spin Hall conductance) could be possible within the same symmetry-breaking phase, meaning that the Laundau classification was insufficient to describe this material [6].

Topological order introduces new observable quantities to characterize phases of matter, going beyond local order parameters and symmetry-breaking. Those quantities are characterized by their robustness to local perturbations and their dependence on the topology of the material (torus, sphere, etc.), hence the name of *topological phases*. Examples of topological quantities include: robust ground-state degeneracy (i.e. degeneracy that does not change when applying local perturbations), non-abelian geometric phase, entanglement, etc.

To understand how those topological phases arise, we first need to introduce a few important concepts. Quantum phases of matter can be defined in two ways: from a Hamiltonian perspective and from a state perspective. Let us start with the Hamiltonian way.

B. Phase transitions from Hamiltonians

Let $\mathcal{H}$ be a set of Hamiltonians, corresponding to the class of systems under consideration (e.g. fermionic/bosonic Hamiltonians, Hamiltonians with a given symmetry, etc.). Within this set, we can define the notion of phase transition.

Definition 1 (Quantum phase transition) *Let $H(g) \in \mathcal{H}$ be a family of Hamiltonians, and $|\psi_0(g)\rangle$ be the ground-state of $H(g)$. We say that a **phase-transition** occurs at g_c if there exists an observable O such that $\langle O \rangle_{|\psi_0(g)\rangle}$ is singular at g_c.*

This definition has the merit of being consistent with the classical definition of phase transitions. However, it is not very practical for our goal of classifying phases, since one needs to consider all possible observables to see if a phase transition occurs at one point. The following theorem gives a more convenient characterization [7]:

Theorem 1 *A phase transition occurs at g_c if and only if the energy gap of $H(g)$ closes at g_c.*

An illustration of this theorem is given in Figure 1.

We can now define a **quantum phase** as an equivalence class of Hamiltonians in $\mathcal{H}$ that can all be connected without going through a phase transition. For instance, the symmetric and symmetry-breaking phases of the Ising model are two examples of quantum phases in the set $\mathcal{H}$ of $\mathbb{Z}_2$-symmetric Hamiltonians (see Figure 2). An important point to notice is that different sets $\mathcal{H}$ will give rise to different phases. Indeed, an unavoidable phase transition within $\mathcal{H}$ can sometimes be avoided by considering the paths outside of $\mathcal{H}$.

The notion of topological order arises when we consider a large class of Hamiltonians, the set $\mathcal{H}_G$ of all gapped Hamiltonians (i.e. whose gap does not close in the thermodynamics limit [8]). Within this set, we can define the concept of topological order[1] [9]:

Definition 2 (Topological order) *A phase in $\mathcal{H}_G$ is said to have topological order if its ground-state degeneracy is stable against any local perturbation*

The quantum error-correcting code known as toric code [10] is an example of topological order. Indeed, it is defined the 4-dimensional stable ground space of a certain Hamiltonian, such that each subspace encodes the logical qubits $|00\rangle$, $|01\rangle$, $|10\rangle$ and $|11\rangle$ with a certain protection against local noise.

To classify all the topologically-ordered phases, it is convenient to switch to another, representation of phase transitions: instead of considering Hamiltonians, we will now consider states.

[1] More precisely, it was noticed in [9] that a better choice to define topological order is the set of *gapped quantum liquid systems*, which behave better in the thermodynamic limit. However, for the purpose of this introduction, we will limit ourselves to gapped quantum systems

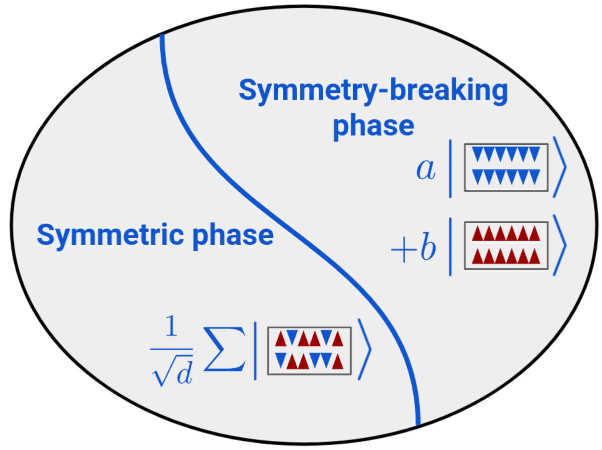


FIG. 2. The two phases of the Ising model, characterized by different symmetry properties, ground state degeneracies and a gap closing at the transition

C. Phase transitions from quantum states

For a set $\mathcal{H}$ of Hamiltonians, we can define $\mathcal{S}$ as all the ground states of Hamiltonians in $\mathcal{H}$. We now say that two states $|\psi(0)\rangle$ and $|\psi(1)\rangle$ in $\mathcal{S}$ are in the same phase if there exists a path $H(g)$ of Hamiltonians, such that $|\psi(0)\rangle$ is the ground state of $H(0)$, $|\psi(1)\rangle$ is the ground state of $H(1)$, and $H(g)$ does not close the gap. This definition, while very natural, still involves Hamiltonians and is therefore not very practical when dealing with states. The following theorem gives a better characterization of phases from a state viewpoint, in the case where $\mathcal{H} = \mathcal{H}_G$ [11]:

Theorem 2 *Two states are in the same phase if and only if they can be connected by a local unitary evolution, i.e. a local quantum circuit of constant-depth.*

The use of quantum information in this characterization suddenly broadens the set of tools that can be used to classify phases of matter. In particular, it gives rise to following corollary:

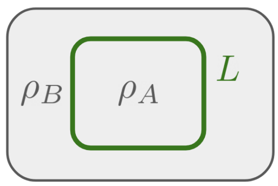
Corollary 1 *There are at least two phases of matter in $\mathcal{H}_G$: one that contains the product state and that we call **trivial order**, and the other that contains some highly-entangled states and that we call **topological order***

Indeed, it is known in quantum information theory that highly-entangled states cannot be created from a product state with a constant-depth local circuit, so there has to be at least two phases. It is shown in [9] that the state definition of topological order is the same as the Hamiltonian definition. The trivial order is said to exhibit **short-range entanglement**, while the topological order has **long-range entanglement** [7].

While the trivial order is made of only one phase by definition (the one that contains the product state), many phases can possibly have topological order. Enumerating all the possible topologically-ordered phases of $\mathcal{H}_G$ for a given dimension will be one of the first goals in our classification.

D. Properties of topological order

So far, we have seen two defining properties of topological order: stable ground-state degeneracy and long-range entanglement. This last characteristic can be quantified more precisely using the notion of **topological entanglement entropy**, discovered by Kitaev and Preskill in 2005 [12]. To define it, let us consider a 2D quantum state $|\psi\rangle$, made of two regions A and B, defining some partial states ρ_A and ρ_B, and separated by a boundary of size L:



A ground state $|\psi\rangle$ is said to *satisfy the area law* if the Von Neumann entropy of ρ_A, $S(\rho_A) = -\text{Tr}[\rho_A \log(\rho_A)]$, is proportional to L (and not the volume of A for instance). It is widely believed that the area law holds for any gapped ground state, and it has been proven in many special cases, e.g. for frustration-free Hamiltonians [13]. In the context of topological order, an alternative form of the area law can be shown:

$$S(\rho_A) \propto \alpha L - \gamma \tag{10}$$

This offset γ defines the topological entanglement entropy, and is for instance equal to $\log(2)$ for the toric code [8, 14]. More generally, any topologically-ordered state is characterized by a non-zero γ [8].

Examples of topologically-ordered states includes fractional quantum Hall states, chiral spin liquids and the toric code [8].

E. Symmetry-protected topological phases

So far, we have only considered phases of gapped Hamiltonians without symmetry ($\mathcal{H}_G$). Therefore, it is natural to ask what happens if we restrict the class of Hamiltonians under consideration by imposing some symmetries. In this case, the trivial order can be separated into multiple phases protected by our symmetry, as illustrated in Figure 3. Those so-called **Symmetry-Protected Topological (SPT) phases**, while characterized by short-range entanglement, are richer and more complicated to classify than topological order. Examples of SPT phases include topological insulators/superconductors and the integer quantum Hall effect. It is worth noticing that symmetry-breaking phases can appear as well, now that our Hamiltonian has some symmetries. In the literature, SPT phases are often defined as preserving the symmetry, distinguishing them from symmetry-breaking phases [8].

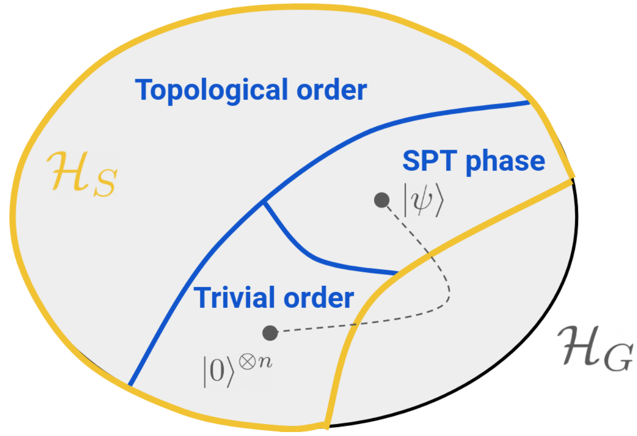


FIG. 3. Illustration of a Symmetry-Protected Topological (SPT) phase: when considering the set $\mathcal{H}_G$ of gapped Hamiltonians without any symmetry constraint, phase transitions in the trivial order can always be avoided by taking another path (dashed line). However, when considering a restricted set of symmetric Hamiltonians $\mathcal{H}_S$, this phase transition can create multiple phases with short-range entanglement called symmetry-protected.

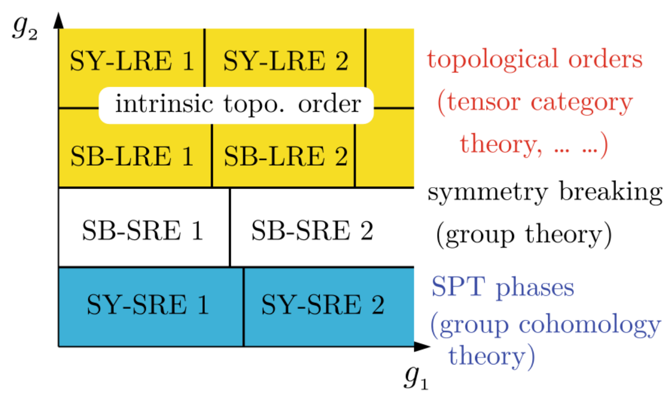


FIG. 4. First layer of classification of topological phases, from [8]. We can first separate short-range entangled (SRE) states from long-range entangled (LRE) states (also called intrinsic topological order). Then, multiple phases can appear within each region, either through symmetry-breaking (SB) or through SPT or SET phases that preserve the symmetry (SY).

Symmetry constraints can also create new phases in the topological order region. When those phases are not simple instances of symmetry-breaking, they are called **Symmetry-Enriched Topological (SET) phases**. Examples of such phases include the topological Mott-insulator and the fractionalized topological insulators [8].

A summary of this first layer of classification can be found in Figure 4

Symmetry				Dimension							
AZ	T	C	S	1	2	3	4	5	6	7	8
A	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AIII	0	0	1	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AI	1	0	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
BDI	1	1	1	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
DIII	-1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
AII	-1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
CII	-1	-1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
C	0	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
CI	1	-1	1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0

FIG. 5. Ten-fold classification of non-interacting systems (i.e. topological insulators/superconductors) [15]. T, C and S are respectively time-reversal, charge-conjugation and chiral symmetry, where $S = T \cdot C$ and $T^2 \in \{0, 1, -1\}$ (same for C^2. For each dimension, $\mathbb{Z}$ indicates the presence of an infinite number of phases and $\mathbb{Z}_2$ the presence of two phases.

IV. CLASSIFICATION OF TOPOLOGICAL PHASES

We have just seen that topological matter can be divided into trivial and topological order, as well as symmetry-enriched, symmetry-protected and symmetry-breaking phases. We already know how to classify symmetry-breaking phases—using the symmetry group of the Hamiltonian and of the ground-state, (G_H, G_ψ)—and would now like to generalize this classification for all the other types of phase. We will now see three special cases where such classification has been performed or where progress has been made: non-interacting systems of boson or fermions, 1D system (complete classification) and higher-dimensional systems (partial classification).

A. Non-interacting systems

One of the first types of system to be fully classified is when $\mathcal{H}$ is made of fermionic or bosonic Hamiltonians with no interaction. Indeed, it can be shown that free fermions and bosons have only ten possible global symmetries, which are all the combinations of time-reversal, charge-conjugation and chiral symmetry. This so-called ten-fold way classification of free bosons/fermions can then be used to describe all possible SPT phases of topological insulators and superconductors, using for instance tools from K-theory [16–18]. An illustration of this classification is given in Figure 5.

Symmetry	No. or Label of different phases
None	1
On-site symmetry of group G (*)	$\omega \in H^2(G, U(1))$
Time Reversal (TR)	2
Translational Invariance (TI)	1
TI + On-site linear symmetry of group G	$\omega \in H^2(G, U(1))$ and $\alpha(G)$
TI + On-site projective symmetry of group G	0
TI + Parity	4
TI + TR	2 if $T^2 = I$ 0 if $T^2 = -I$

FIG. 6. Summary of the classification of 1D boson/spin systems, from [8]

B. One-dimensional systems

A complete classification of one-dimensional systems was established independently by two groups in 2011 [7, 19]. This classification contains two important results:

1. There is no topological order in 1D (all ground-states have short-range entanglement)
2. The SPT phases are labelled by (G_H, G_ψ, ω) where $\omega \in H^2(G_\psi, U(1))$ is a projective representation of G_ψ, living in the second cohomology group $H^2(G_\psi, U(1))$. Figure 6 gives a summary of all the different symmetries and labels for bosonic systems.

To derive those results, the authors of both [7] and [19] used two key ideas: 1) all 1D ground-states satisfy the area law and can therefore be written as matrix-product states (MPS) [20] ; 2) generalized local unitary evolution (where degrees of freedom are allowed to be removed) also preserves the phase. Using those key ideas, proving for instance point 1 simply consists in finding a renormalization procedure that drives any MPS to a product state.

C. Towards higher dimensions

Finally, significant progress has been made to classify higher-dimensional systems. For instance, 2D systems with topological order, with a unique ground state degeneracy, or with a PEPS ground state (generalization of MPS in 2D), can be classified using projective representations, similarly to the 1D case [19]. In the general 2D case, the mathematical formalism becomes much more involved. For instance, 2D bosonic systems can be labelled by modular tensor categories [6] and fermionic systems by fusion categories [21].

In 3D, a classification has been established in several cases: when point-like excitations are all bosonic [22], when some point-like excitations are fermions [23] and when the symmetries behave similarly to topological insulators and superconductors [24].

V. BEYOND TOPOLOGICAL ORDER?

In the same way that Landau's symmetry-breaking paradigm was extended by considering topological phases of matter, the reader might rightly wonder whether some phases exist beyond topological order. In other words, are there phases that cannot be described using the framework developed above?

The closest candidate to a new type of order is the **fracton phase of matter** [25]. Discovered by Jeongwan Haah in 2011 [26], fractons can only appear in three-dimensional systems and are characterized by a set of unique properties, that make them different from ordinary topological matter: immobile quasi-particles, exploding ground-state degeneracy with the system size, fractal structure (hence the name "fracton"), etc. While fractons appeared for the first time in a particular model, the X-Cube model (also known as Haah's code), it was later discovered (2015) that they can emerge in a large variety of ways and form their own new type of matter [27, 28].

The question of how to extend topological order to include fractons is still an open research problems, but several candidates for a new general theory have been proposed, such as cellular topological states [29] or homogeneous topological order [30]. Meanwhile, fracton phases of matter have been characterized by the notion of "foliated fracton order" [31, 32], where the three-dimensional system is decomposed into 2D layers, and the addition or suppression of layers is allowed in the adiabatic evolution. Whether the final theory of fractons will require only some small tweaks to the topological framework, or will lead to a completely novel understanding of matter is an essential question whose resolution could lead to fascinating applications.

VI. DISCUSSION

While it seems hard to summarize all topological phases in a single periodic table as in chemistry, this case study has allowed us to take a glance at different components of such a table. We saw that symmetry-breaking phases can be classified by the symmetry group of the ground-state and the Hamiltonian, that topological phases separate a topological from a trivial order through entanglement and ground-state degeneracy properties, and that for symmetric Hamiltonians, both orders are made of many sub-categories, mainly the symmetry-protected and the symmetry-enriched phases. Enumerating all those phases for different types of Hamiltonians with different symmetries is one of the main challenges of the field, but significant progress has been made in last ten years: a complete characterization of 1D systems (using projective representations) and non-interacting systems (using K-theory) has been achieved while only a few steps remain for a classification of higher-dimensional systems in most cases of interest. Meanwhile, the discovery of fractons has challenged the whole field by introducing a completely new type of 3D matter that resists the current topological classification, but steady progress has been made to develop a framework that would include them.

I can identify four open research directions that could have important consequences for the future of the field:

1. Finishing and simplifying the classification of topological phases in 2D and 3D systems (and potentially higher dimensions)

2. Developing a coherent framework of topological order that include fractons

3. Classifying gapless states: so far we have only considered gapped states, but many interesting quantum and topological effect seem to happen in gapless systems [33].

4. Pursuing the connections between quantum error-correction and topological phases of matter. For instance, could we use fractons as a practical quantum code, as proposed in [34]?

Finally, our last challenge is experimental: could we bridge the gap between our very formal classification and the discovery and design of new materials? Topological matter has a bright future with an endless list of potential applications, and understanding them better through classification is certainly a first step towards their practical implementation.

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